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Control over a Bandwidth Limited Signal to Noise Ratio constrained Communication Channel

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Abstract— **Stabilisability of a minimum phase unstable continuous plant is studied under the presence of a bandwidth limited and Signal to Noise Ratio constrained communication link. The problem is addressed in two different ways: first through the use of an LTI filter explicitly modelling the bandwidth limitation, and in second place, for the case of one real unstable pole, through the Poisson Integral Formula and design attenuation requirements on the power outside the assigned bandwidth. Results show that when a bandwidth limitation is in existence this increases the minimum value of Signal to Noise Ratio required for stabilisability. An example is used to study both approaches.**

I. INTRODUCTION

Feedback control over communication links has become an area of growing interest in recent years with works such as, for example, [1], [2], [3], [4] and [5]. See also [6] and the references therein.

Generally, the communication link involves some pre- and post-processing of the signals that are sent through a communication channel, for example, filtering, analog-to-digital (A-D) conversion, coding, modulation, decoding, demodulation and digital-to-analog (D-A) conversion.

In this paper we neglect all pre- and post- signal processing involved in the communication link, which is thus reduced to the communication channel itself and, as in [7], [8] and [9], or more recently [10], we model the communication channel as an additive white Gaussian noise (AWGN) channel, but with the added fundamental feature of limited bandwidth. This bandwidth constraint may be imposed, for example, to avoid interference between different channels in a communication system.

Of the two possible configurations for the location of the idealised communication channel (measurement path and control path), we consider the case of a communication channel over the control link. Such a setting is common in practice and arises, for example, when actuators are far from the controller and have to communicate through a (perhaps partially wireless) communication network. Nonetheless in an LTI setting both forms are equivalent, and it is a simple matter to restate the results for the case of where measurement is performed over a communication channel.

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Stabilisability of the resulting feedback loop has been studied in relation to quantisation, bit rate limitations, bandwidths constraint or time delays over the communication channel. As another example, a different line of investigation is pursued in [11], [12] and [13], which make use of topological and entropy concepts. This article, instead, follows the line of research developed in [7], [8], [9] and [10] using signal to noise ratio constraints.

These works model the communication channel through the idealisation of an AWGN channel, see for example [14], imposing a power constraint on the signal that has been sent. The line of investigation developed in [7], [8] and [9] has also been successfully linked to the topological results in [11], including the effects of non-minimum phase zeros and time delays in the plant, both in the continuous and discrete setting, with output feedback and state-space feedback.

We also consider an AWGN channel for the communication channel, as for example in [7]. The Signal to Noise Ratio (SNR) constraint is achieved by transferring the original power constraint $\mathcal{P}$ on the signal u_s to the signal u_f, as in Figure 1. We assume $|F(j\omega)| \leq 1$, for simplicity. The bandwidth constraint is modelled through the low pass transfer function F.

In this paper only the continuous time output feedback case is treated; however, extensions to the discrete case as well as to the state feedback case (both continuous and discrete) should follow in a similar fashion to [7], when we are dealing with a minimum phase unstable plant with no time delays.

The main result of this work is an expression for the minimum SNR required to guarantee stabilisability when we face the case of a AWGN communication channel with an assigned bandwidth. The obtained SNR bound proves to be more demanding than the bound previously obtained in [7] for a AWGN channel with infinite bandwidth. For the case in which the plant has one unstable real pole and $F(s)$ is restricted due to a second power constraint defined for the range of frequencies outside the assigned bandwidth, a necessary condition is developed by restating the main result.

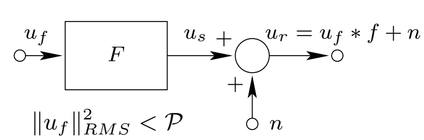

Fig. 1. Additive white Gaussian noise channel with power and bandwidth constraint

The paper is organised as follows: in Section II we briefly review a linear time invariant framework for control over a bandwidth limited and SNR constrained communication channel; Section III addresses design issues involving power attenuation outside the assigned bandwidth. Section IV presents concluding remarks. All proofs are listed in the Appendix.

II. An LTI Framework for Control over a Communication Link

Consider a linear, band limited, AWGN channel with an input power constraint as depicted in Figure 1. The channel output signal is given (in the Laplace transform domain, for continuous-time systems) by

$$U_r(s) = F(s)U_f(s) + N(s),$$

where the filter $F(s)$ represents the channel transfer function (limited bandwidth) and $n(t)$ is a zero-mean white Gaussian noise with intensity Φ, that is,

$$E\{n(t)\} = 0, \quad E\{n(t)n(t+\tau)\} = \Phi\,\delta(t-\tau),$$

where $E\{\cdot\}$ denotes the expectation operation. Assuming the closed loop is exponentially stable, the distribution of $u_s(t)$ converges to a stationary stochastic process with root mean square (RMS) value

$$\|u_f\|_{RMS} = \left(E\{u_f(t)^2\}\right)^{1/2}.$$

The power of a continuous-time stationary stochastic process u_f is defined as $\|u_f\|^2_{RMS}$, which can be alternatively expressed (e.g.,pp. 21–22, [15]) in terms of the *power spectral density* $S_{u_f}(\omega)$,

$$\|u_f\|^2_{RMS} = \frac{1}{2\pi}\left[\int_{-\infty}^{\infty} S_{u_f}(\omega)\,d\omega\right]. \tag{1}$$

The power constraint in the channel model of Figure 1 is represented by requiring that the power of $u_f(t)$ be bounded by some predetermined positive value $\mathcal{P}$,

$$\|u_f\|^2_{RMS} < \mathcal{P}$$

Such a channel model is widely used in Communications (e.g., [16]; [14]; [17]), and is also useful to represent, to some extent, effects of roundoff and quantisation in A-D and D-A converters [18].

On using the channel model of Figure 1 we obtain the LTI feedback loop of Figure 2, in which $P(s)$ and $C(s)$ respectively are the transfer functions of the plant and the controller, and $y(t)$ is the output of the system.

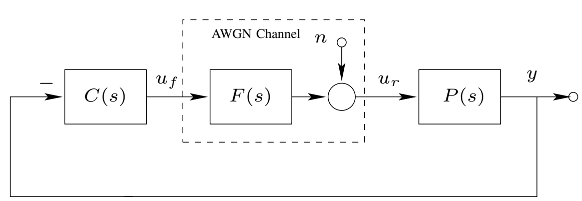

Fig. 2. Simplified continuous-time feedback loop over a control plus noise corrupted channel communication

We assume that the controller $C(s)$ is such that the feedback loop of Figure 2 is asymptotically stable. We also assume that the plant $P(s)$ is proper and minimum phase (it does not contain either zeros in $\mathbb{C}^+$ or time delays), although it may be unstable, and that the filter $F(s)$ is proper, stable and minimum phase. The results here can be extended to cover NMP or time delays though with significantly more involved calculations.

Since the closed loop system is asymptotically stable, the control signal received by the plant $u_f(t)$ resulting from the input noise $n(t)$ is a stationary stochastic process with Gaussian distribution. It is well known that the power spectral density of $u_f(t)$ can be expressed as

$$S_{u_f}(\omega) = T_F(j\omega)S_n(\omega)T_F(-j\omega), \tag{2}$$

where $T_F(s)$ is the closed loop transfer function between $n(t)$ and $u_f(t)$ in Figure 2, that is,

$$T_F(s) = -\frac{P(s)C(s)}{1+P(s)C(s)F(s)}, \tag{3}$$

and

$$S_n(\omega) = \int_{-\infty}^{\infty} E\{n(\tau)n(t+\tau)\}e^{-j\omega\tau}\,d\tau = \Phi,$$

is the power spectral density of $n(t)$. Hence, by virtue of (1),

$$\|u_f\|^2_{RMS} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[T_F(j\omega)T_F(-j\omega)\right]\Phi\,d\omega = \|T_F\|^2_{H_2}\Phi, \tag{4}$$

where $\|T_F\|_{H_2} = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty}|T_F(j\omega)|^2\,d\omega}$ is the H_2 norm of $T_F(s)$. Note that $\|T_F\|_{H_2}$ is finite because $T_F(s)$ is stable and strictly proper. To find the lowest achievable value of $\|u_f\|_{RMS}$ we have to find the lowest achievable value of $\|T_F\|_{H_2}$ over the class of all stabilising controllers.

If the plant is unstable, $\|T_F\|_{H_2}$ has a positive lower bound that cannot be further reduced by any choice of the controller, as we show in the following lemma. We denote by $\mathcal{K}$ the set of stabilising proper LTI controllers $C(s)$ for $P(s)$.

Proposition 1: Consider the feedback loop of Figure 2. Assume that the plant $P(s)$ is proper and minimum phase, and has m poles $p_k, k = 1, 2, \ldots, m$ in $\mathbb{C}^+$, and that $C(s)$ is such that the closed-loop is asymptotically stable. Then an SNR which suffices in guaranteeing stabilisability must satisfy:

$$\frac{\mathcal{P}}{\Phi} > \inf_{C\in\mathcal{K}} \|T_F\|^2_{H_2} = \sum_{k=1}^{m} 2Re\{p_k\}\,|R_k(p_k)|^2 \tag{5}$$

where

$$R_k(s) = \frac{\left(s+\overline{p}_{k-1}\right)R_{k-1}(s) - 2Re\{p_{k-1}\}R_{k-1}(p_{k-1})}{s-p_{k-1}}, \tag{6}$$

for $k = 2, \ldots, m$, and

$$R_1(p_k) = F^{-1}(p_k) \quad k = 1, \ldots, m. \tag{7}$$

Proof: See Appendix. ∎

Corollary 1: Under the assumptions of Proposition 1, if the filter has the same magnitude at all the unstable poles of $P(s)$ are real, i.e., $|F(p_k)| = f_0, \forall k = 1, \ldots, m$, then:

$$\frac{\mathcal{P}}{\Phi} > \inf_{C \in \mathcal{K}} \|T_F\|_{H_2}^2 = \frac{1}{f_0^2} \sum_{k=1}^{m} 2Re\{p_k\} \tag{8}$$

In order to better interpret the main result reported in Proposition 1 an example follows.

Example 1: Consider a plant $P(s)$ with an unstable real pole located at $p = 1$ and also consider a Chebyshev low pass filter of order 6 and cut-off frequency ω_B, as a potential candidate for the role of filter F (throughout this example ω_B and bandwidth will be considered equivalent concepts). Chebyshev filters also have a parameter R which represents the level of attenuation between in band and out band frequencies.

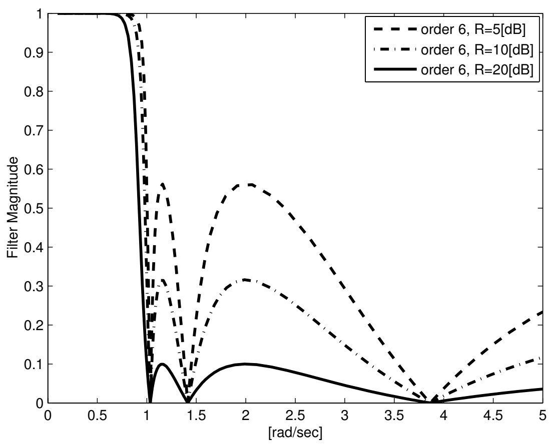

Fig. 3. Filter magnitude response for the case of Chebyshev filters of order 6 with $R = 5$ [dB], dashed line, $R = 10$ dB, dash-dot line, and $R = 20$ [dB] solid line for $\omega_B = 1[rad/sec]$.

Figure 3 reports the filter magnitude for a range of frequencies up to $5[rad/sec]$. Different selections for R show how the out band attenuation changes.

The SNR bound that suffices to guarantee stabilisability in the case of a Chebyshev filter of order 6 is reported in Figure 4, with values of target attenuation of $R = 20$ [dB], a less stringent attenuation of 10 [dB] and an even lesser attenuation of 5 [dB]. The main observation is that for $R = 5$ [dB] we will have less attenuation than for $R = 10$ [dB] or $R = 20$ [dB]. Less attenuation implies a smaller SNR value, due to the inverse proportionality between the magnitude of the filter and the SNR bound, mostly in the initial range of values for ω_B. The effect of the difference in the values of R disappears as ω_B increases, since the filter magnitude at the unstable pole location is approaching 1, no matter what filter attenuation is implied by R. Note that the feature of initial increase in the SNR, when ω_B is approximately $0.1[rad/sec]$, is due to the selection of Chebyshev filters for F and decreases in range when R increases. Indeed if we select Butterworth filters instead of Chebyshev filters, this feature disappears (not shown).

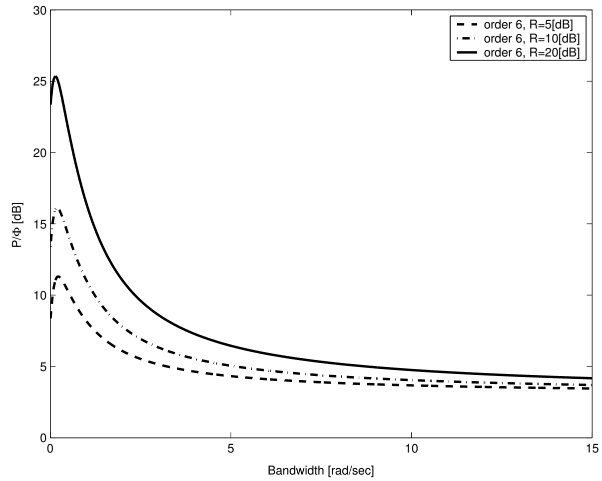

Fig. 4. SNR bound for the case of a Chebyshev filter of order 6 designed to achieve 5 [dB] attenuation for $\omega > \omega_B$, dashed line, 10 [dB] attenuation, dash-dot line, and 20 [dB] attenuation, solid line (unstable pole located at 1).

The main assumption behind results presented in Figure 4 is that we have a candidate for the filter F that will define the bandwidth of our communication channel, but this may be not always the case or we may be interested in other aspects of the problem that not involve explicitly the filter.

Note 1: As might be expected from the problem formulation, the expression in equation (5) is invariant under all possible orderings of the m unstable poles $p_k, k = 1, 2, \cdots, m$, for a proof see [19].

Note 2: please note that the definition of $R_k(s)$ in equation (6) is well defined. An inductive argument, not presented here, can be used to prove this. Also note that for the particular case of $p_k = p_{k-1}$, $R_k(p_k)$ in (6) will become:

$$R_k(p_k) = R_{k-1}(p_k) + 2Re\{p_k\} \left. \frac{dR_{k-1}(s)}{ds} \right|_{s=p_k} \tag{9}$$

III. Out of band Power attenuation requirement

The presence of a filter limiting the communication link bandwidth can also be seen as an assigned bandwidth for the channel itself. Naturally design requirements, apart from those of bandwidth, will also have to specify a power level, $\mathcal{P}_{out}$ say, for the communication channel to comply with at frequencies greater than ω_B, in order to minimise interference to other potential users, as for example, in a frequency division multiplexing scheme.

To quantify the effect of a constraint on the level of out of band interference on the required SNR for closed loop stability, we impose a general profile on the filter frequency response as:

$$|F(j\omega)| \leq \epsilon, \quad \forall \omega > \omega_B \tag{10}$$

The level of attenuation ϵ outside the communication bandwidth can be interpreted as the admissible ratio of out of band interference power ($\mathcal{P}_{out}$) relative to the total power ($\mathcal{P}$). Indeed, we can approximate the out of band power of

u_s as follow:

$$\frac{1}{\pi}\int_{\omega_B}^{\infty} S_{u_s}(\omega)d\omega \leq \sup_{\omega>\omega_B} |F(j\omega)|^2 \frac{1}{\pi}\int_{\omega_B}^{\infty} S_{u_f}(\omega)d\omega$$
$$\leq \epsilon^2\mathcal{P} = \mathcal{P}_{out} \tag{11}$$

Proposition 2: Consider a plant $P(s)$ proper and minimum phase with one pole $p \in \mathbb{R}^+$. The SNR requirement necessary for stabilisability in terms of the ratio between the power constraints inside and outside the assigned bandwidth, $\mathcal{P}/\mathcal{P}_{out}$, is given by:

$$\frac{\mathcal{P}}{\Phi} \geq 2p\left(\frac{\mathcal{P}}{\mathcal{P}_{out}}\right)^{\frac{\pi-2\arctan(\omega_B/p)}{\pi}} \tag{12}$$

Proof: Consider the Poisson Integral Formula, see [20], for the case of one unstable real pole p :

$$\log|F(p)| = \frac{1}{\pi}\int_{-\infty}^{\infty}\log|F(j\omega)|\frac{p}{p^2+\omega^2}d\omega \tag{13}$$

It is not difficult to see that the following inequality holds true for the magnitude of any filter F that is contained by its idealised version described in equation (10).

$$|F(p)|^{-1} \geq \left(\frac{1}{\epsilon}\right)^{\frac{\pi-2\arctan(\omega_B/p)}{\pi}} \tag{14}$$

Replacing equation (14) into (5) and noting, from equation (11), that $1/\epsilon^2 = \mathcal{P}/\mathcal{P}_{out}$ ends the proof. ■

The result in Proposition 2 expose the relationship between the SNR necessary for stabilisability and the power ratio requirement, $\mathcal{P}/\mathcal{P}_{out}$. Also, it presents the interplay that exists between the available bandwidth, ω_B, and the unstable pole location p. If the bandwidth tends to zero we will have that the required SNR will be inversely proportional to the power attenuation, whilst if the bandwidth tends to infinity the result for one unstable pole with no band limitation obtained in [7] will be recovered. Again an example will help to better interpret the result.

Example 2: Consider a plant $P(s)$ with one unstable real pole. For the case of $p = 1$, in Figure 5, values of 10, 1000 and 10^5 for the ratio $\mathcal{P}/\mathcal{P}_{out}$ have been studied, as per equation (12).

The resulting SNR bounds show that the greater the out of band power attenuation required, the greater the estimated SNR will be.

As an example see that at $\omega_B = 5[rad/sec]$ the estimate SNR is 4 [dB], for an out of band power constraint 10 times smaller than the in band power constraint. The SNR estimate increases to 7 [dB], for an out of band attenuation of 1000, and it jumps to almost 10 [dB], for an out of band attenuation of 10^5. This can be seen in terms of the filter F as follows: the greater the out-band power attenuation requirement the more stringent the requirement for the communication channel to confine the transmitted signal inside the assigned bandwidth, i.e the smaller the value of ϵ will have to be for that filter (as per equation (11)).

It is not surprising, therefore, that increasing the factor $\mathcal{P}/\mathcal{P}_{out}$ or increasing the attenuation requirement for the cut-off frequency of the filter $F(s)$ have the equivalent effect of increasing the SNR lower bound for stabilisability. Note in any case that the lower bound defined in equation (12) is necessary to guarantee stabilisability, whilst the expression in equation (5) is sufficient. Also note that all bounds presented in Figures 4 and 5 tend to the same limit as ω_B increases. This limit is given by the result obtained in [7] for the infinite bandwidth AWGN channel and for this specific example is given by:

$$\frac{\mathcal{P}}{\Phi}[dB] \geq 10\log_{10}(2p) \tag{15}$$

A comparison of the results portrayed in Figures 4 and 5 with equation (15), obtained in [7], shows that the presence of a band limitation in the communication channel increases the value of SNR required for stabilisability.

Finally to investigate the effect of the pole location over the bound defined as in equation (12), a factor of 1000 has been selected for $\mathcal{P}/\mathcal{P}_{out}$. The result in figure 6 confirms the intuition that the position of the unstable pole heavily affects the required SNR value for stabilisability. However, this same penalty is reduced if the bandwidth is increased. See for example that for $p = 1$ and $\omega_B = 1$ [rad/sec] the SNR value is 18 [dB], for $\omega_B = 5$ [rad/sec] decreases to 7 [dB] and for $\omega_B = 10$ [rad/sec] diminishes to 5 [dB].

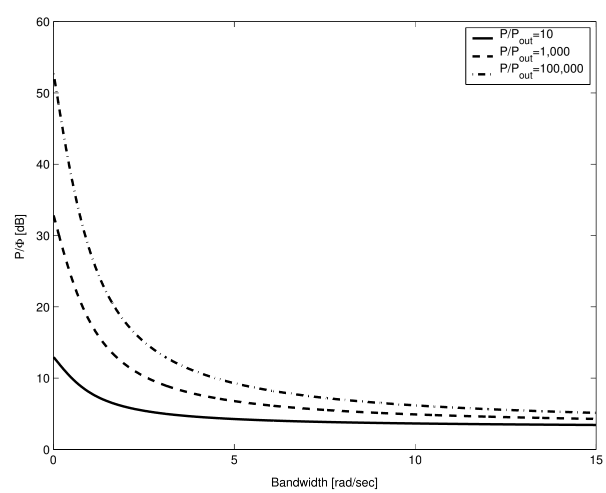

Fig. 5. Estimated SNR necessary bound for $\mathcal{P}/\mathcal{P}_{out}$ equal to 10, solid line, 1000, dashed line, and 10^5, dash dotted line.

Note 3: A similar result can be obtained for the case of two unstable real poles, p_1 and p_2, and is reported here as

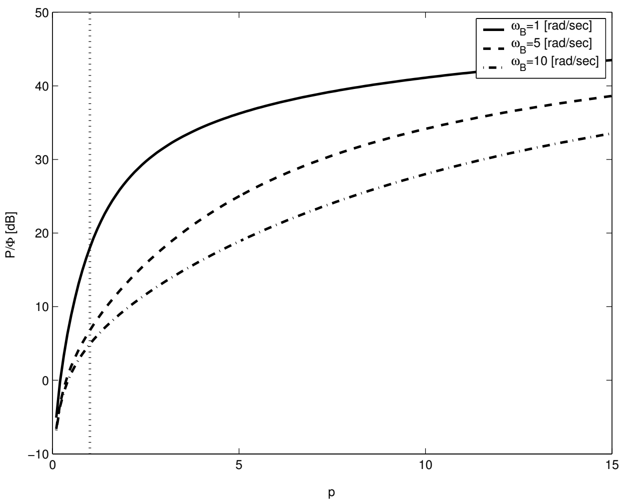

Fig. 6. Estimated SNR necessary bound for $\mathcal{P}/\mathcal{P}_{out}$=1000 with $\omega_B = 1$ [rad/sec], solid line, $\omega_B = 5$ [rad/sec], dashed line and $\omega_B = 10$ [rad/sec], dash-dotted line. Unstable pole location varying from 0 to 15.

an example on how to extend this ideas.

$$\frac{\mathcal{P}}{\Phi} \geq 2p_1 \left(\frac{p_1+p_2}{p_2-p_1}\right)^2 \left(\frac{\mathcal{P}}{\mathcal{P}_{out}}\right)^{\frac{\pi-2\arctan(\omega_B/p_1)}{\pi}} - \frac{8p_1p_2\,(p_1+p_2)}{(p_2-p_1)^2}\left(\frac{\mathcal{P}}{\mathcal{P}_{out}}\right)^{\frac{\pi-\arctan(\omega_B/p_1)-\arctan(\omega_B/p_2)}{\pi}} + 2p_2\left(\frac{p_1+p_2}{p_2-p_1}\right)^2\left(\frac{\mathcal{P}}{\mathcal{P}_{out}}\right)^{\frac{\pi-2\arctan(\omega_B/p_2)}{\pi}} \tag{16}$$

It is possible to include as many real unstable poles as needed, but with the associated cost of increasingly more demanding algebra. The case of $p_1 = p_2$ is not addressed in equation (16), for that would require the use of equation (9) and the Poisson Integral Formula, instead of equation (7), twice, for p_1 and p_2 and the Poisson Integral Formula.

IV. Conclusions and Remarks

In this paper the case of a sufficient SNR requirement for stabilisability over a band limited communication channel has been studied. The main result shows the interplay between the real parts of the unstable poles of the plant and the bandwidth available for the communication channel. The SNR requirement for stabilisability will be more demanding if the unstable poles of the plant are located outside the bandwidth for the communication channel. Vice versa, if the unstable poles of the plant are well inside the bandwidth of the communication channel the SNR requirement approaches the result for an infinite bandwidth channel.

For one real unstable pole in the plant the filtered version of the communication channel creates a sufficient SNR lower bound for stabilisability inversely proportional to the square of the filter magnitude evaluated at that pole. The obtained bound is greater than the result presented in [7] for an infinite bandwidth AWGN channel. An out of band power attenuation requirement has been also studied and results suggest that a design problem can be stated in this framework. This alternative definition for the SNR lower bound will consider out of band power attenuation requirements and it will be necessary for stabilisability.

Future work includes band limited SNR constrained communication channels for discrete systems, as well as the case of state feedback. Multivariable systems application will follow.

V. Appendix

To compute the minimum value of $\|T\|^2_{H_2}$ over the class of all stabilising controllers, we apply a technique used in [21]. We start by deriving an expression for T based on a parametrisation of all stabilising controllers. Represent $F(s)P(s)$ by a co-prime factorisation

$$F(s)P(s) = \frac{N(s)}{B_p(s)},$$

where $N(s) \in RH_\infty$ (the space of proper, stable, rational functions), and

$$B_p(s) = \prod_{k=1}^{m} \frac{s-p_k}{s+\overline{p}_k}$$

is the Blaschke product of all poles of $P(s)$ in $\overline{\mathbb{C}}^+$. By using the well-known Youla controller parametrisation [22], we can represent any stabilising controller for the feedback loop in Figure 2 by[1]

$$C = \frac{X+B_pQ}{Y-NQ}, \tag{17}$$

where Q, X and Y are in RH_∞, with X and Y satisfying the Bezout identity

$$NX + B_pY = 1. \tag{18}$$

By replacing (17) in (18), we find that T_F can be expressed as

$$T_F = (1 - B_p(Y-NQ))\,F^{-1} = N(X+B_pQ)F^{-1}, \tag{19}$$

where the last equality follows from the Bezout identity. Thus, from (19), the problem of finding the lowest value of $\|T_F\|^2_{H_2}$ over the class of stabilising controllers reduces to that of finding

$$\inf_{Q\in RH_\infty} \left\|N(X+B_pQ)F^{-1}\right\|_{H_2}. \tag{20}$$

The rest of the proof involves a sequential partial fraction expansion of the unstable poles of $P(s), p_k, k = 1, 2, \dots, m$, that will render the minimisation in (20) trivial, and leave as a remainder the RHS of (5).

For simplicity, and to start a recursive argument, define

$$R_1 = N(X+B_pQ)F^{-1}. \tag{21}$$

Now let p_1 be the first unstable pole of $P(s)$. Since the factor $\frac{s+\overline{p}_1}{s-p_1}$ is all-pass, i.e., $\left|\frac{j\omega+\overline{p}_1}{j\omega-p_1}\right| = 1$, we can write

$$\|R_1\|^2_{H_2} = \left\|\left(\frac{s+\overline{p}_1}{s-p_1}\right)R_1\right\|^2_{H_2} = \left\|\frac{r_1}{s-p_1}+R_2\right\|^2_{H_2} \tag{22}$$

[1]Dependency on s is suppressed to simplify notation when convenient.

where

$$R_2 = \left(\frac{s+\overline{p}_1}{s-p_1}\right) R_1 - \frac{r_1}{s-p_1}, \tag{23}$$

and r_1 is the residue of $\left(\frac{s+\overline{p}_1}{s-p_1}\right) R_1$ at $s = p_1$, i.e.,

$$\begin{aligned} r_1 &= \lim_{s\to p_1} (s+\overline{p}_1) R_1 \\ &= 2Re\{p_1\} R_1(p_1). \end{aligned} \tag{24}$$

Notice from (23) and (24) that we have just obtained (6) for $k = 2$. Now, $R_2 \in H_2$, since its residue at $s = p_1$ is zero, while $\frac{r_1}{s-p_1} \in H_2^{\perp}$. Therefore, we can rewrite (20) as

$$\inf_{Q\in RH_{infty}} \|R_1\|_{H_2}^2 = \left\|\frac{r_1}{s-p_1}\right\|_{H_2^{\perp}}^2 + \inf_{Q\in RH_\infty} \|R_2\|_{H_2}^2 . \tag{25}$$

By continuing the same procedure with the rest of the unstable poles of $P(s)$, we obtain, for $k = 1, \dots, m$, the recursive formula

$$R_{k+1} = \left(\frac{s+\overline{p}_k}{s-p_k}\right) R_k - \frac{r_k}{s-p_k}, \tag{26}$$

where r_k is the residue of $\left(\frac{s+\overline{p}_k}{s-p_k}\right) R_k$ at $s = p_k$, i.e.,

$$\begin{aligned} r_k &= \lim_{s\to p_k} (s+\overline{p}_k) R_k \\ &= 2Re\{p_k\} R_k(p_k). \end{aligned} \tag{27}$$

Equations (26) and (27) yield (6) for all k. Equation (7) follows by noting that

$$R_1(p_k) = N(p_k)X(p_k)F^{-1}(p_k) = F^{-1}(p_k),$$

since, from (18), $N(p_k)X(p_k) = 1 - B_p(p_k)Q(p_k) = 1$.

By construction $R_k \in H_2 \forall k = 1, \dots, m$, which gives

$$\inf_{Q\in RH_\infty} \|R_1\|_{H_2}^2 = \sum_{k=1}^{m} \left\|\frac{r_k}{s-p_k}\right\|_{H_2^{\perp}}^2 + \inf_{Q\in RH_\infty} \|R_{m+1}\|_{H_2}^2 . \tag{28}$$

We claim that $\inf_{Q\in RH_\infty} \|R_{m+1}\|_{H_2}^2 = 0$. Indeed, note that

$$\begin{aligned} R_{m+1} &= B_p^{-1} R_1 - \sum_{k=1}^{m-1} \frac{r_k}{s-p_k} \prod_{i=k}^{m} \left(\frac{s+\overline{p}_i}{s-p_i}\right) - \frac{r_m}{s-p_m} \\ &= \left\{ B_p^{-1} NXF^{-1} - \sum_{k=1}^{m-1} \frac{r_k}{s-p_k} \prod_{i=k}^{m} \left(\frac{s+\overline{p}_i}{s-p_i}\right) - \frac{r_m}{s-p_m} \right\} \\ &\quad + NQF^{-1} \end{aligned}$$

Because R_{m+1} and NQF^{-1} are stable, the term between braces in the last equality is also stable. Therefore, given any $\varepsilon > 0$, there exists a $Q_\varepsilon \in RH_\infty$ such that $\|R_{m+1}\|_{H_2} < \varepsilon$, proving that $\inf_{Q\in RH_\infty} \|R_{m+1}\|_{H_2}^2 = 0$.

Finally then,

$$\inf_{Q\in RH_\infty} \|R_1\|_{H_2}^2 = \sum_{k=1}^{m} \left\|\frac{r_k}{s-p_k}\right\|_{H_2^{\perp}}^2 . \tag{29}$$

Since

$$\left\|\frac{r_k}{s-p_k}\right\|_{H_2^{\perp}}^2 = \frac{|r_k|^2}{2Re\{p_k\}} = 2Re\{p_k\} |R_k(p_k)|^2 ,$$

which follows from (27), then (29) and (4) yield (5), completing the proof.

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